Discrete Differential Geometry and Integrable Systems

March 21, 2018

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$$\text{Discrete Differential Geometry} \quad \xrightarrow{\text{continuous limit}} \quad \text{Differential Geometry}$$

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What is the best way?

One possible way is to discretize taking into account the symmetry properties of the problem.

Discretizing Lagrangian equations

Let $L : TQ = (q, \dot{q}) \to \mathbb{R}$ be a Lagrangian function.

$$S[c] = \int_0^1 L(c(t), \dot{c}(t))dt\,, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

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The discrete functional is

$$S_D = \sum_k L(q_k, q_{k+1})\,, \quad q_k = q(t_k)$$

(over a discrete trajectory) and the discrete Euler-Lagrange equation is

$$d_1 L(q_k, q_{k+1}) + d_2 L(q_{k-1}, q_k) = 0$$

Example: falling objects

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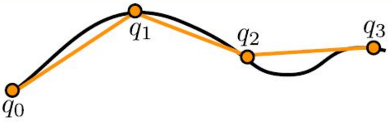

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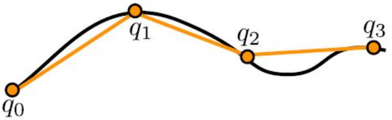

$$(q_{k+1} - q_k) - (q_k - q_{k-1}) = q_{k+1} - 2q_k + q_{k-1} = -g$$

Discretizing a surface

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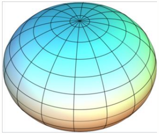

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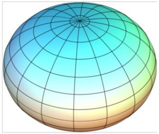

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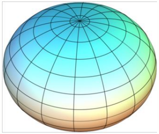

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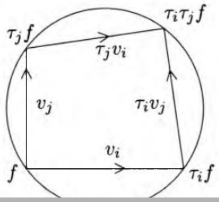
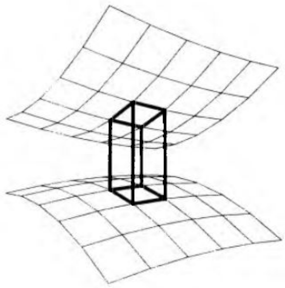
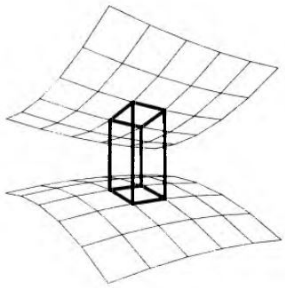
- It is called *circular* if any elementary quadrilaterals is inscribed in a circle





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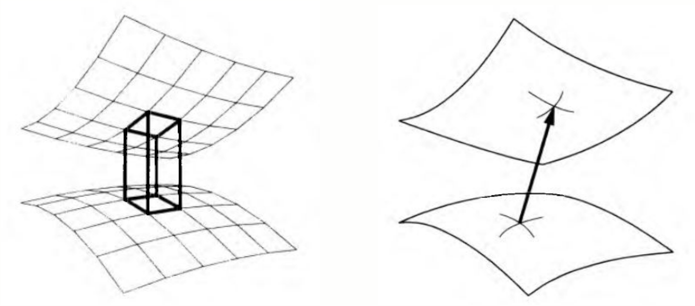

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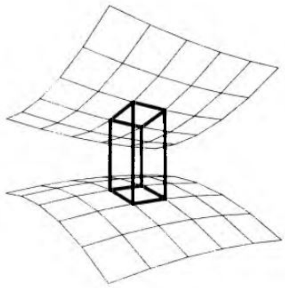

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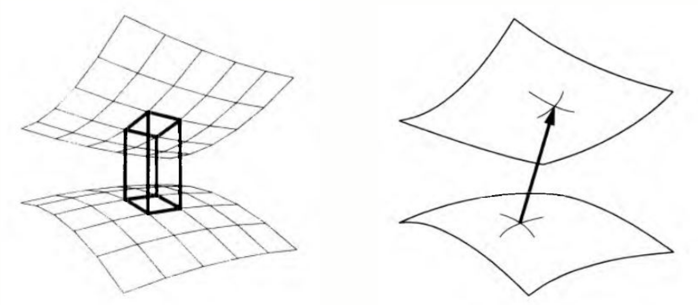

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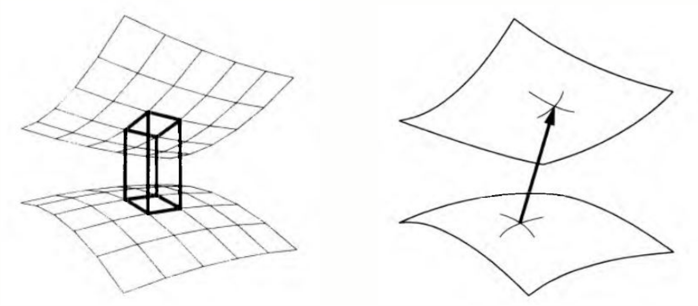

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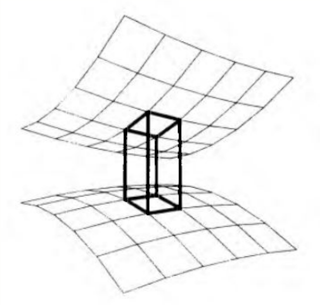 

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Let H be the mean curvature and K the Gaussian curvarture. It is well known that $H^2 - K$ is a conformal invariant.

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One can try to discretize both the mean and Gaussian curvature.

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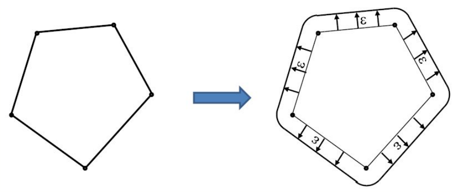

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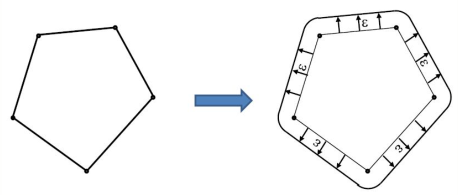

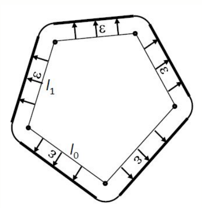

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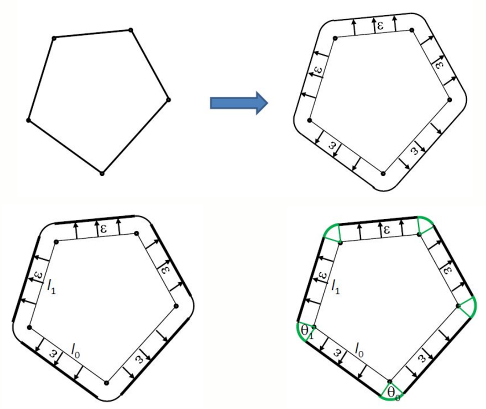

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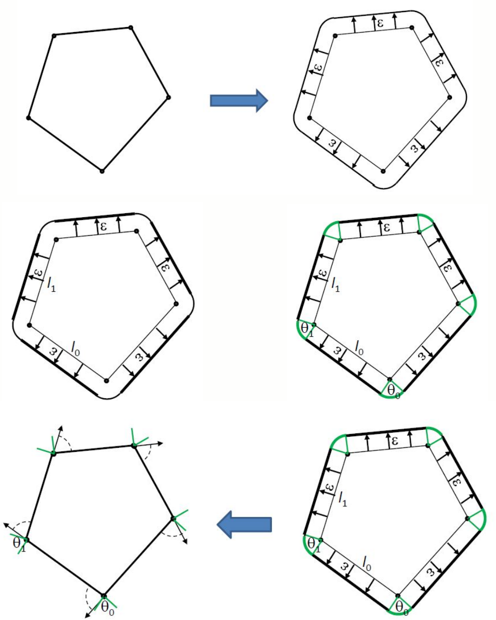

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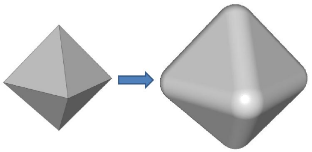

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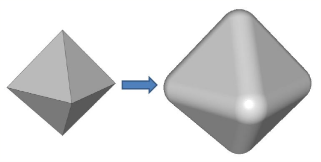

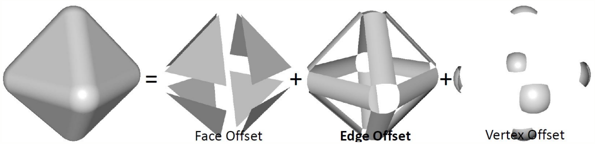

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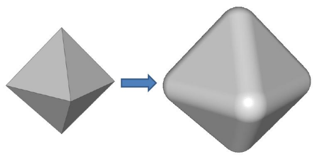

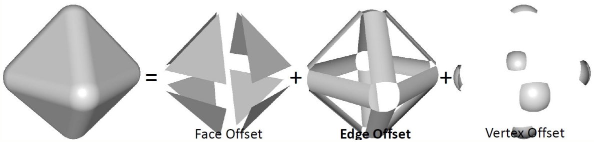

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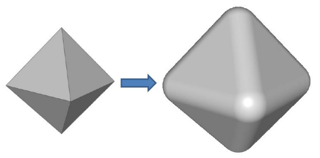

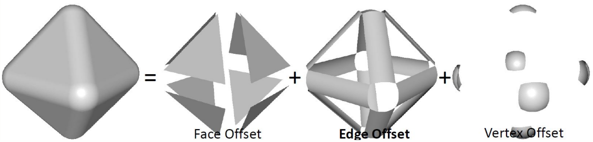

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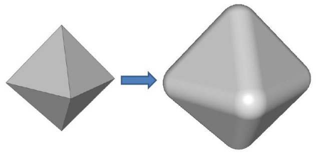

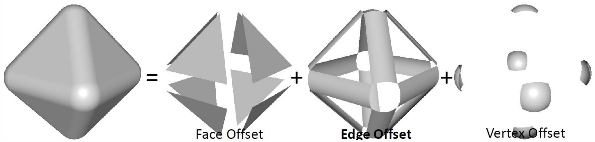

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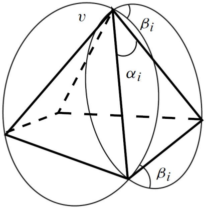

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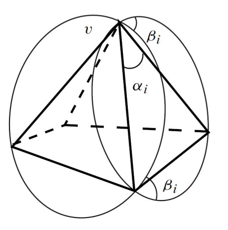

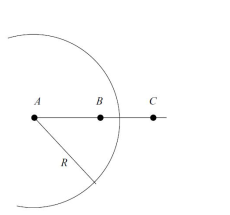

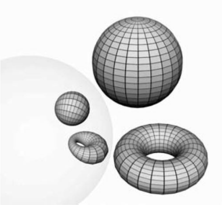

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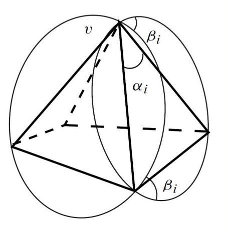

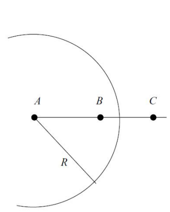

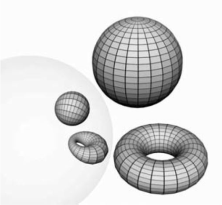

The functional

$$\mathcal{W}(v) = \sum_{e \ni v} \beta(e) - 2\pi$$

over all edges incident to v is conformal invariant.

$$\mathcal{W}(S) = \sum_{v \in Vertices} \mathcal{W}(v)$$